

GENERATORS OF THE EISENSTEIN–PICARD MODULAR GROUP IN THREE COMPLEX DIMENSIONS

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ABSTRACT. Little is known about the generators system of the higher dimensional Picard modular groups. In this paper, we prove that the higher dimensional Eisenstein–Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$ in three complex dimensions can be generated by eight given transformations by using a similar method described in Falbel et.al [Proc. Amer. Math. Soc. 139: 2439–2447, 2011].

1. INTRODUCTION

As the complex hyperbolic analogue of Bianchi groups $\mathbf{PSL}(2; \mathcal{O}_d)$, Picard modular groups are $\mathbf{SU}(n, 1; \mathcal{O}_d)$ where \mathcal{O}_d is the ring of algebraic integers of the imaginary quadratic extension $\mathbf{Q}(i\sqrt{d})$ for any positive square free integer d (see [8]). The elements of the ring \mathcal{O}_d can be described (see [9]):

$$\mathcal{O}_d = \begin{cases} \mathbf{Z}[i\sqrt{d}] & \text{if } d \equiv 1, 2 \pmod{4} \\ \mathbf{Z}[\frac{1+i\sqrt{d}}{2}] & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

It is well known that the ring \mathcal{O}_d is Euclidean for positive square free integer d if and only if $d = 1, 2, 3, 7, 11$; see [11]. In particular, if $d = 3$, $\mathcal{O}_d = \mathbb{Z}[\omega_3]$, where $\omega_3 = (-1 + i\sqrt{3})/2$. Picard modular groups are the simplest arithmetic lattices in $\mathbf{SU}(n, 1)$.

There are many results on Picard modular groups in two complex dimensions. In [6], Falbel and Parker constructed a remarkable simple fundamental domain of the Eisenstein–Picard modular group $\mathbf{SU}(2, 1; \mathbb{Z}[\omega_3])$. Applying Poincaré polyhedra theorem, they showed that $\mathbf{SU}(2, 1; \mathbb{Z}[\omega_3])$ admits a presentation with two generators. Similarly, in [1], they obtained a presentation of the Gauss–Picard modular group $\mathbf{SU}(2, 1; \mathcal{O}_1)$. Francsics and Lax also independently obtained the generators of the Gauss–Picard modular group acting on the two-dimensional complex hyperbolic space (see [3, 4, 5]).

However, constructing explicit fundamental domains in complex hyperbolic space is much more difficult than in real hyperbolic space. Therefore, it is interesting to look for another method to get generators system of Picard modular groups. In [2], Falbel et al. gave a simple algorithm to obtain the generators of the Gauss–Picard modular group. Wang et al. [12] showed that this algorithm can also be extended to the Eisenstein–Picard modular group $\mathbf{SU}(2, 1; \mathbb{Z}[\omega_3])$. But it is still open for other Picard modular groups. Recently, Zhao [13] obtained the generators of the Euclidean Picard modular groups $\mathbf{SU}(2, 1; \mathcal{O}_d)$ for $d = 2, 7, 11$ by using

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a different method. We note that very little is known about the geometric and algebraic properties, e.g., explicit fundamental domains, generators, presentations of the higher dimensional Picard modular groups $\mathbf{SU}(n, 1; \mathcal{O}_d)$.

In the present paper, we prove that the method of [2] can also be applied to the higher dimensional Eisenstein–Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$ and obtain a simple description in terms of generators. More precisely, we prove that the Eisenstein–Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$ can be generated by eight transformations, four Heisenberg translations, three Heisenberg rotations and an involution.

This paper is organized as follows. Section 2 gives a brief introduction to complex hyperbolic geometry and the Picard modular group. The main results and its proofs are included in Section 3.

2. PRELIMINARIES

In this section, we recall some basic materials in complex hyperbolic geometry and Picard modular group. The general reference on these topics are [7, 10].

Let $\mathbb{C}^{n,1}$ denote the vector space \mathbb{C}^{n+1} equipped with the Hermitian form

$$\langle \mathbf{w}, \mathbf{z} \rangle = z_1 \overline{w_{n+1}} + z_2 \overline{w_2} + \dots + z_n \overline{w_n} + z_{n+1} \overline{w_1}$$

where \mathbf{w} and \mathbf{z} are the column vectors in $\mathbb{C}^{n,1}$ with entries $z_1, z_2, \dots, z_n, z_{n+1}$ and $w_1, w_2, \dots, w_n, w_{n+1}$ respectively. Equivalently, we may write

$$\langle \mathbf{w}, \mathbf{z} \rangle = \mathbf{z}^* J \mathbf{w}$$

where \mathbf{w}^* denote the Hermitian transpose of \mathbf{w} and

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Define the subgroups V_-, V_0 of $\mathbb{C}^{n,1}$ as follows

$$V_- = \{\mathbf{v} \in \mathbb{C}^{n,1} : \langle \mathbf{v}, \mathbf{v} \rangle < 0\},$$

$$V_0 = \{\mathbf{v} \in \mathbb{C}^{n,1} : \langle \mathbf{v}, \mathbf{v} \rangle = 0\}.$$

Define a right projection map \mathbb{P} from the subspace of $\mathbb{C}^{n,1}$ consisting of those \mathbf{z} with $z_{n+1} \neq 0$ to \mathbb{C}^n by

$$\mathbb{P} : \begin{pmatrix} z_1 \\ \vdots \\ z_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} z_1/z_{n+1} \\ \vdots \\ z_n/z_{n+1} \end{pmatrix}.$$

The complex hyperbolic n -space $\mathbf{H}_{\mathbb{C}}^n = \mathbb{P}(V_-) \subset \mathbb{C}^n$. This is a paraboloid in \mathbb{C}^n , called the Siegel domain.

The boundary of the Siegel domain model of $\mathbf{H}_{\mathbb{C}}^n$ consists of those points in $\mathbb{P}(V_0)$ together with a distinguished point at infinity, which denote ∞ . The finite points in the boundary of $\mathbf{H}_{\mathbb{C}}^n$ naturally carry the structure of the generalized Heisenberg group (denoted by \mathcal{H}_{2n-1}), which is defined to $\mathbb{C}^{n-1} \times \mathbb{R}$ with the group law

$$(\xi, \nu) \cdot (z, u) = (\xi + z, \nu + u + 2\Im \langle \xi, z \rangle).$$

Here $\langle \xi, z \rangle = z^* \xi$ is the standard positive defined Hermitian form on \mathbb{C}^{n-1} . In particular, we write $\|\xi\|^2 = \xi^* \xi$.

Motivated by this, we define horospherical coordinates on complex hyperbolic space. To each point $(\xi, \nu, u) \in \mathcal{H}_{2n-1} \times \mathbb{R}_+$, we associated a point $\psi(\xi, \nu, u) \in V_-$. Similarly, ∞ and each point $(\xi, \nu, 0) \in \mathcal{H}_{2n-1} \times \{0\}$ is associated to a point in V_0 by ψ . The map ψ is given by

$$\psi(\xi, \nu, u) = \begin{pmatrix} (-|\xi|^2 - u + i\nu)/2 \\ \xi \\ 1 \end{pmatrix}, \quad \psi(\infty) = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix}.$$

We also define the origin 0 to be the point in $\partial \mathbf{H}_{\mathbb{C}}^n$ with horospherical coordinates $(0, 0, 0)$. We have

$$\psi(0) = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 1 \end{pmatrix}.$$

The holomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^n$ is the group $\mathbf{SU}(n, 1)$ of complex linear transformations, which preserve the above Hermitian form. That is

$$\mathbf{SU}(n, 1) \equiv \{G \in \mathbf{SL}(n+1, \mathbb{C}) : \langle Gz, Gw \rangle = \langle z, w \rangle\}.$$

The corresponding matrix $G = (g_{jk})_{i,j=1}^n$ satisfies the following condition

$$(1) \quad G^* J G = J,$$

where G^* denote the conjugate transposition of the matrix G . The determinant of the matrix G is normalized to be equal to 1.

The Picard modular groups $\mathbf{SU}(n, 1; \mathcal{O}_d)$ are holomorphic automorphism subgroups of $\mathbf{H}_{\mathbb{C}}^n$ defined as

$$\mathbf{SU}(n, 1; \mathcal{O}_d) \equiv \{G \in \mathbf{SL}(n+1, \mathbb{C}) : G^* J G = J, g_{ij} \in \mathcal{O}_d\}.$$

We now discuss the decomposition of complex hyperbolic isometries. We begin by considering those elements fixing 0 and ∞ .

The matrix group $\mathbf{U}(n-1)$ acts by Heisenberg rotation. In horospherical coordinates, the action of $U \in \mathbf{U}(n-1)$ is given by

$$(\xi, \nu, u) \mapsto (U\xi, \nu, u).$$

The corresponding matrix in $\mathbf{SU}(n, 1)$ acting on $\mathbb{C}^{n,1}$ is

$$M_U \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The positive real numbers $r \in \mathbb{R}^+$ act by Heisenberg dilation. In horospherical coordinates, this acting is given by

$$(\xi, \nu, u) \mapsto (r\xi, r^2\nu, r^2u).$$

In $\mathbf{SU}(n, 1)$ the corresponding matrix is

$$A_r \equiv \begin{pmatrix} r & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1/r \end{pmatrix}.$$

The Heisenberg group acts by Heisenberg translation. For $(\tau, t) \in \mathcal{H}_{2n-1}$, this is

$$N_{(\tau, t)}(\xi, \nu) = (\tau + \xi, t + \nu + 2\Im\langle \tau, \xi \rangle).$$

As a matrix $N_{(\tau,t)}$ is given by

$$N_{(\tau,t)} \equiv \begin{pmatrix} 1 & -\tau^* & (-\|\tau\|^2 + it)/2 \\ 0 & I_{n-1} & \tau \\ 0 & 0 & 1 \end{pmatrix}.$$

Heisenberg translations, rotations and dilations generate the Heisenberg similarity group. This is the full subgroup of $\mathbf{SU}(n, 1)$ fixing ∞ .

Finally, there is one more important acting, called an inversion R , which interchanges 0 and ∞ . In matrix notation this map is

$$R \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let Γ_∞ be the stabilizer subgroup of ∞ in $\mathbf{SU}(n, 1)$. That is

$$\Gamma_\infty \equiv \{g \in \mathbf{SU}(n, 1) : g(\infty) = \infty\}.$$

Lemma 2.1. *Let $G = (g_{jk})_{j,k=1}^4 \in \mathbf{SU}(3, 1)$. Then $G \in \Gamma_\infty$ if and only if $g_{41} = 0$.*

Using Langlands decomposition, any element $P \in \Gamma_\infty$ can be decomposed as a product of a Heisenberg translation, dilation, and a rotation:

$$(2) \quad P = N_{(\tau,t)} A_r M_U = \begin{pmatrix} r & -\tau^* U & (-\|\tau\|^2 + it)/2r \\ 0 & U & \tau/r \\ 0 & 0 & 1/r \end{pmatrix}.$$

The parameters satisfy the corresponding conditions.

3. THE MAIN RESULT AND ITS PROOF

In this section we extend the method in [2] to the higher dimensional Eisenstein-Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$.

Let $\mathbf{U}(2; \mathbb{Z}[\omega_3])$ be the unitary group $\mathbf{U}(2)$ over the ring $\mathbb{Z}[\omega_3]$. Recall that the unitary matrix $A \in \mathbf{U}(2)$ is of the following form

$$\mathbf{U}(2) = \{A = \begin{pmatrix} a & b \\ -\lambda \bar{b} & \lambda \bar{a} \end{pmatrix} : |\lambda| = 1, |a|^2 + |b|^2 = 1\}.$$

Then we can see that the elements in $\mathbf{U}(2; \mathbb{Z}[\omega_3])$ are of the following form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$$

where $a, b = \pm 1, \pm \omega_3, \pm \omega_3^2$.

It is easy to find that

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b = \pm 1, \pm \omega_3, \pm \omega_3^2 \right\}$$

can be generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & -\omega_3 \end{pmatrix}, \begin{pmatrix} -\omega_3 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}.$$

Therefore we have the following result.

Lemma 3.1. $\mathbf{U}(2; \mathbb{Z}[\omega_3])$ can be generated by the following three unitary matrixes

$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, U_2 = \begin{pmatrix} 1 & 0 \\ 0 & -\omega_3 \end{pmatrix}, U_3 = \begin{pmatrix} -\omega_3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Next, we consider the subgroup Γ_∞ of the Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$.

Lemma 3.2. Let $\Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$ denote the subgroup Γ_∞ of Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$. Then any element $P \in \Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$ if and only if the parameters in the Langlands decomposition of P satisfy the conditions

$$r = 1, t \in \sqrt{3}\mathbb{Z}, \tau = (\tau_1, \tau_2)^T \in \mathbb{Z}[\omega_3]^2, U \in \mathbf{U}(2; \mathbb{Z}[\omega_3])$$

the integers $t/\sqrt{3}$ and $\|\tau\|^2$ have the same parity.

Proof. Let $P \in \Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$ be the Langlands decomposition form (2). Then it is easy to see that $r = 1$, $t \in \sqrt{3}\mathbb{Z}$ and $U \in \mathbf{U}(2; \mathbb{Z}[\omega_3])$. Since the entries τ_1, τ_2 of τ and the entry $(-\|\tau\|^2 + it)/2$ are in the ring $\mathbb{Z}[\omega_3]$, we get that $t/\sqrt{3} \in \mathbb{Z}$ and $|\tau_1|^2 + |\tau_2|^2 \in \mathbb{Z}$. Further more, they have the same parity. \square

Proposition 3.1. Let $\Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$ be stated as above. Then $\Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$ is generated by the Heisenberg translations $N_1 = N_{((1,0)^T, \sqrt{3})}$, $N_2 = N_{((\omega_3, 0)^T, \sqrt{3})}$, $N_3 = N_{((0,1)^T, \sqrt{3})}$, $N_4 = N_{((0, \omega_3)^T, \sqrt{3})}$, and the Heisenberg rotations M_{U_i} ($i = 1, 2, 3$).

Proof. Suppose $P \in \Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$. According to Lemma 3.2, there is no dilation component in its Langlands decomposition, that is

$$P = N_{(\tau, t)} M_U = \begin{pmatrix} 1 & -\tau^* & (-\|\tau\|^2 + it)/2 \\ 0 & I_2 & \tau \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the unitary matrix $U \in \mathbf{U}(2; \mathbb{Z}[\omega])$. Then the rotation component of P in the Langlands decomposition is generated by M_{U_i} ($i = 1, 2, 3$) by Lemma 3.1.

We now consider the Heisenberg translation part of P , $N_{(\tau, t)}$. Let

$$\tau = (a_1 + b_1\omega_3, a_1 + b_1\omega_3)^T,$$

where $a_1, b_1, a_2, b_2 \in \mathbb{Z}$. Since $\tau \in \mathbb{Z}[\omega_3]^2$. Then $N_{(\tau, t)}$ splits as

$$\begin{aligned} N_{(\tau, t)} &= N_{((a_1 + b_1\omega_3, a_1 + b_1\omega_3)^T, t)} \\ (3) \quad &= N_{((a_1, 0)^T, \sqrt{3}a_1)} \circ N_{((b_1\omega_3, 0)^T, \sqrt{3}b_1)} \circ N_{((0, a_2)^T, \sqrt{3}a_2)} \circ N_{((0, b_2\omega_3)^T, \sqrt{3}b_2)} \\ &\quad \circ N_{((0, 0)^T, t + \sqrt{3}a_1b_1 - \sqrt{3}a_1 - \sqrt{3}b_1 + \sqrt{3}a_2b_2 - \sqrt{3}a_2 - \sqrt{3}b_2)}. \end{aligned}$$

Here the Heisenberg translations

$$N_{((a_1, 0)^T, \sqrt{3}a_1)}, N_{((b_1\omega_3, 0)^T, \sqrt{3}b_1)}, N_{((0, a_2)^T, \sqrt{3}a_2)}, N_{((0, b_2\omega_3)^T, \sqrt{3}b_2)}$$

can be written as follows

$$\begin{aligned} N_{((a_1, 0)^T, \sqrt{3}a_1)} &= N_{((1, 0)^T, \sqrt{3})}^{a_1}, \\ N_{((b_1\omega_3, 0)^T, \sqrt{3}b_1)} &= N_{((\omega_3, 0)^T, \sqrt{3})}^{b_1}, \\ (4) \quad N_{((0, a_2)^T, \sqrt{3}a_2)} &= N_{((0, 1)^T, \sqrt{3})}^{a_2}, \\ N_{((0, b_2\omega_3)^T, \sqrt{3}b_2)} &= N_{((0, \omega_3)^T, \sqrt{3})}^{b_2}. \end{aligned}$$

We claim that the number

$$\left(t - \sqrt{3}(-a_1b_1 + a_1 + b_1 - a_2b_2 + a_2 + b_2) \right) / 2\sqrt{3}$$

is an integer, namely,

$$\frac{t}{\sqrt{3}} - (-a_1b_1 + a_1 + b_1 - a_2b_2 + a_2 + b_2) \in 2\mathbb{Z}.$$

According to Lemma 3.2, the integers $t/\sqrt{3}$ and

$$(5) \quad \begin{aligned} \|\tau\|^2 &= |a_1 + b_1\omega_3|^2 + |a_2 + b_2\omega_3|^2 \\ &= a_1^2 - a_1b_1 + b_1^2 + a_2^2 - a_2b_2 + b_2^2 \end{aligned}$$

have the same parity. It can be easily seen that

$$\begin{aligned} &a_1^2 - a_1b_1 + b_1^2 + a_2^2 - a_2b_2 + b_2^2 + (-a_2b_2 + a_2 + b_2) + (-a_1b_1 + a_1 + b_1) \\ &= a_1(a_1 + 1) + b_1(b_1 + 1) + a_2(a_2 + 1) + b_2(b_2 + 1) - 2a_1b_1 - 2a_2b_2 \in 2\mathbb{Z}. \end{aligned}$$

Hence $t/\sqrt{3}$ and $-a_1b_1 + a_1 + b_1 - a_2b_2 + a_2 + b_2$ have the same parity. This proves that

$$t_1 = \frac{t - \sqrt{3}(-a_1b_1 + a_1 + b_1 - a_2b_2 + a_2 + b_2)}{2\sqrt{3}} \in \mathbb{Z}.$$

Therefore the Heisenberg translation

$$N_{((0,0)^T, t + \sqrt{3}a_1b_1 - \sqrt{3}a_1 - \sqrt{3}b_1 + \sqrt{3}a_2b_2 - \sqrt{3}a_2 - \sqrt{3}b_2)}$$

can be written as

$$N_{((0,0)^T, 2\sqrt{3})}^{t_1}.$$

The Heisenberg translation $N_{((0,0)^T, 2\sqrt{3})}$ can be generated by $N_{((1,0)^T, \sqrt{3})}, N_{((0,1)^T, \sqrt{3})}^{-1}$ and M_1 , that is

$$N_{((0,0)^T, 2\sqrt{3})} = (N_{((1,0)^T, \sqrt{3})} \circ N_{((0,1)^T, \sqrt{3})}^{-1} \circ M_1)^2.$$

This proposition is proved. \square

Now we prove our main result.

Theorem 3.2. *The Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$ is generated by four Heisenberg translations $N_j (j = 1, 2, 3, 4)$, three Heisenberg rotations $M_{U_i} (i = 1, 2, 3)$ and the involution R .*

Proof. Let $G = (g_{jk})_{j,k=1}^4$ be an element of the group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$. We only need to consider $G \notin \Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$. According to Lemma 2.1, we have $g_{41} \neq 0$. G maps the point ∞ to the point $(g_{11}/g_{41}, g_{21}/g_{41}, g_{31}/g_{41})$. Since $G(\infty)$ is in $\partial\mathbf{H}_{\mathbb{C}}^3$, then

$$(6) \quad 2\Re\left(\frac{g_{11}}{g_{41}}\right) = -\left|\frac{g_{21}}{g_{41}}\right|^2 - \left|\frac{g_{31}}{g_{41}}\right|^2$$

Consider the Heisenberg translation $N_{G(\infty)}$ that maps $(0, 0)$ to $G(\infty)$. The translation $N_{G(\infty)}$ is not necessary in the Picard modular group $\mathbf{SU}(3, 1; \mathbb{Z}[\omega_3])$ except if $|g_{41}| = 1$. However, we know that

$$RN_{G(\infty)}^{-1}G = P.$$

So we will successively approximate $N_{G(\infty)}^{-1}$ by Heisenberg translations in the Picard modular group to decrease the value $|g_{41}|^2 \in \mathbb{Z}$ until it becomes 0. Then G can be expressed as a product of the generators in $\Gamma_\infty(3, 1; \mathbb{Z}[\omega_3])$ and R . Since the ring $\mathbb{Z}[\omega_3]$ is Euclidean, this approximation process has finitely many steps.

Next we calculate the entry in the lower left corner of the product

$$\begin{aligned}
 G_1 &= RN_{(\tau,t)}G \\
 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\bar{\tau}_1 & -\bar{\tau}_2 & (-\|\tau\|^2 + it)/2 \\ 0 & 1 & 0 & \tau_1 \\ 0 & 0 & 1 & \tau_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} G \\
 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & \tau_1 \\ 0 & 0 & -1 & \tau_2 \\ 1 & -\bar{\tau}_1 & -\bar{\tau}_2 & (-\|\tau\|^2 + it)/2 \end{pmatrix} G
 \end{aligned} \tag{7}$$

It follows that the entry $g_{41}^{(1)}$ of G_1 is equal to

$$\begin{aligned}
 g_{41}^{(1)} &= g_{11} - g_{21}\bar{\tau}_1 - g_{21}\bar{\tau}_2 + g_{41} \frac{-\|\tau\|^2 + it}{2} \\
 &= g_{41} \left(\frac{g_{11}}{g_{41}} - \frac{g_{21}}{g_{41}}\bar{\tau}_1 - \frac{g_{31}}{g_{41}}\bar{\tau}_2 + \frac{-\|\tau\|^2 + it}{2} \right) \\
 &= -g_{41} \left[\left(-\Re \left(\frac{g_{11}}{g_{41}} \right) + \Re \left(\frac{g_{21}}{g_{41}}\bar{\tau}_1 \right) + \Re \left(\frac{g_{31}}{g_{41}}\bar{\tau}_2 \right) + \frac{\|\tau\|^2}{2} \right) \right. \\
 &\quad \left. - i \left(\Im \left(\frac{g_{11}}{g_{41}} \right) - \Im \left(\frac{g_{21}}{g_{41}}\bar{\tau}_1 \right) - \Im \left(\frac{g_{31}}{g_{41}}\bar{\tau}_2 \right) + \frac{t}{2} \right) \right] \\
 &= -g_{41}(I_1 - iI_2).
 \end{aligned} \tag{8}$$

Using (6), we can simplify I_1 to

$$I_1 = \frac{1}{2} \left(\left| \frac{g_{21}}{g_{41}} + \tau_1 \right|^2 + \left| \frac{g_{31}}{g_{41}} + \tau_2 \right|^2 \right).$$

Let $\frac{g_{21}}{g_{41}} = x_1 + iy_1$, $\frac{g_{31}}{g_{41}} = x_2 + iy_2$, $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Note that

$$\tau = (a_1 + b_1\omega_3, a_2 + b_2\omega_3)^t = (a_1 - \frac{b_1}{2} + \frac{b_1\sqrt{3}i}{2}, a_2 - \frac{b_2}{2} + \frac{b_2\sqrt{3}i}{2})^t,$$

where $a_1, b_1, a_2, b_2 \in \mathbb{Z}$. We can select four appropriate integers a_1, b_1, a_2 and b_2 satisfying

$$\left| x_1 + (a_1 - \frac{b_1}{2}) \right| \leq \frac{1}{2}, \quad \left| y_1 + \frac{b_1\sqrt{3}i}{2} \right| \leq \frac{\sqrt{3}}{4}$$

and

$$\left| x_2 + (a_2 - \frac{b_2}{2}) \right| \leq \frac{1}{2}, \quad \left| y_2 + \frac{b_2\sqrt{3}i}{2} \right| \leq \frac{\sqrt{3}}{4}.$$

Hence, we obtain the upper bound

$$|I_1| \leq \frac{1}{2} \left(\left(\frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{4} \right)^2 + \left(\frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{4} \right)^2 \right) = \frac{14}{32}.$$

Since $t/\sqrt{3} \in \mathbb{Z}$ and $|\tau_1|^2 + |\tau_2|^2 \in \mathbb{Z}$ have the same parity, we can get the inequality

$$|I_2| = \left| \Im \left(\frac{g_{11}}{g_{41}} \right) - \Im \left(\frac{g_{21}}{g_{41}}\bar{\tau}_1 \right) - \Im \left(\frac{g_{31}}{g_{41}}\bar{\tau}_2 \right) + \frac{t}{2} \right| \leq \frac{\sqrt{3}}{2}$$

by selecting some t in I_2 . Therefore, we have the estimation of $g_{41}^{(1)}$

$$|g_{41}^{(1)}|^2 = |g_{41}|^2 |I_1 + iI_2|^2 = |g_{41}|^2 (I_1^2 + I_2^2) \leq |g_{41}|^2 \left[\left(\frac{14}{32} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 \right] = \frac{241}{256} |g_{41}|^2.$$

We can reduce the matrix of the transformation G to the matrix of a transformation G_n with $g_{41}^{(n)} = 0$ by repeating this approximation procedure finitely many times. According to Lemma 2.1, this condition implies that the G_n belongs to the subgroups Γ_∞ . As we shown in Proposition 3.1, the subgroup Γ_∞ can be generated by the Heisenberg translations $N_j (j = 1, 2, 3, 4)$ and the Heisenberg rotations $M_{U_i} (i = 1, 2, 3)$. Since the approximation procedure just uses the transformations in Γ_∞ and involution R . Hence the proof of Theorem 3.2 is completed. \square

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